

**ON THE EVOLUTION OF MARRIAGE FUNCTIONS:
IT TAKES TWO TO TANGO***

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Summary

In earlier work, we characterized two-sex marriage functions as multiplicative perturbations of the Ross solution, that is, heterosexually random or proportionate mixing. These perturbations are expressed in terms of the preferences/affinities of males for females and viceversa. Male and female preferences/affinities are not independent, in fact, they depend on the availability of male and female behavioral “genotypes”. The key result of this article says that knowledge of the preferences/affinities of one gender completely characterizes the preferences/affinities of both genders; in other words, it takes two to tango. This is the basic content of the T^3 Theorem. In addition, we show that different sets of preferences/affinities, that is, distinct behavioral “genotypes”, can give rise to identical mixing/mating probabilities, the determinants of the behavioral “phenotypes”. Hence, different sets of individual decisions can lead to identical social dynamics—a fact well established in genetics. The work in this article extends and applies prior results published in this journal.

1. Introduction

Marriage functions are solutions to the two-sex mixing/pairing problem. Despite their importance in areas such as population genetics (mating functions), demography (population projection), cultural anthropology (preservation and dissemination of cultural traits), and evolutionary biology (life history), their application has been quite limited. Most researchers have addressed theoretical issues in these areas through the use of single-sex models or highly simplified two-sex models. A basic premise that it is being ignored is that “it takes two individuals to tango”. The difficulties involved can be seen in the pioneering work of Kendall¹, Keyfitz², Fredrickson³, McFarland⁴, Parlett⁵, and Pollard⁶.

Over the last few years, we (Busenberg and Castillo-Chavez) developed an axiomatic framework that allows for the systematic study of marriage functions⁷⁻⁹. Although our work is still in its infancy, it has already been applied in areas as diverse as cultural anthropology¹⁰, demography¹¹, epidemiology and food web dynamics¹², and parameter estimation¹³⁻¹⁵. We provide a detailed characterization of marriage functions for populations defined through fixed characteristics such as race, language, biological species, religion, level of education, and socio-economic level. Therefore, the framework described in this article can be utilized to address sociological and biological questions for this type of populations via finite dimensional deterministic or stochastic models. Questions about populations defined in terms of dynamic characteristics such as age require the use of infinite dimensional systems that, in principle, are more difficult to analyze¹⁶.

We⁹ characterized two-sex marriage functions as multiplicative perturbations of the Ross solution, that is, heterosexually random or proportionate mixing. These perturbations are defined in terms of the preferences/affinities of males for females and viceversa. Male and female preferences are not independent, in fact, they depend on the availability of male and female behavioral “genotypes”. The key result of this article says that knowledge of the preferences/affinities of one gender completely characterizes the preferences/affinities of both genders; in other words, it takes two to tango--the basic content of the T^3 Theorem. In this article, we also show that different sets of preferences/affinities, that is, distinct behavioral “genotypes”, can give rise to identical mixing/mating probabilities which

are the determinants of the behavioral “phenotypes”. Hence, different sets of individual decisions can lead to identical social dynamics—a fact well established in genetics. The work in this article extends prior results published in this journal¹⁷.

This article is organized as follows: Section 2 reviews and summarizes our earlier work on marriage functions using fixed classifications; Section 3 introduces flexible parametric families of mixing solutions that make connections to data possible and presents an example using our own data to estimate a heterosexual mixing matrix; Section 4 discusses the relationship between males and females preferences/affinities through the T^3 theorem; Section 5 summarizes our results and suggests future directions.

2. Two-sex Mixing Framework for a Population with Fixed Characteristics

We consider a population with L types of males and N types of females. Let $T_i^m(t)$ denote the number of males of type i at time t ($i=1, \dots, L$), and $T_j^f(t)$ denote the number of females of type j at time t ($j=1, \dots, N$). In addition, we let c_i ($i=1, \dots, L$) and b_j ($j=1, \dots, N$) denote the rates, assumed constant (for simplicity), of pair formation for males of type i and for females of type j , respectively.

A two-sex marriage function is described by two matrices: $P(t)=\{p_{ij}(t)\}$ and $Q(t)=\{q_{ji}(t)\}$, where $p_{ij}(t)$ denotes the probability that a male of type i pairs with a female of type j given that he has formed a heterosexual partnership at time t , and $q_{ji}(t)$ denotes the probability that a female of type j pairs with a male of type i given that she has formed a heterosexual partnership at time t . The pair $(P(t), Q(t))$ is called a marriage or mixing/pair-formation matrix if and only if it satisfies the following properties at all times:

$$(A1) \quad 0 \leq p_{ij}(t) \leq 1 \text{ and } 0 \leq q_{ji}(t) \leq 1 \text{ for } i=1, \dots, L, j=1, \dots, N.$$

$$(A2) \quad \sum_{j=1}^N p_{ij}(t) = 1 \text{ for } i=1, \dots, L; \quad \sum_{i=1}^L q_{ji}(t) = 1 \text{ for } j=1, \dots, N.$$

$$(A3) \quad c_i T_i^m(t) p_{ij}(t) = b_j T_j^f(t) q_{ji}(t) \text{ for } i=1, \dots, L, j=1, \dots, N.$$

$$(A4) \quad p_{ij}(t) \equiv q_{ji}(t) \equiv 0 \text{ by definition if } c_i b_j T_i^m(t) T_j^f(t) = 0 \text{ for some } i, 1 \leq i \leq L, \text{ and/or} \\ \text{some } j, 1 \leq j \leq N.$$

Property (A3) expresses the fact that the total average rate of pair formation between males of type i and females of type j must be equal, while property (A4) asserts that individuals from populations that do not interact cannot possibly mix. An immediate consequence of the above properties is that the total average rates of male and female activity must agree at all times, that is,

$$\sum_{i=1}^L c_i T_i^m = \sum_{j=1}^N b_j T_j^f. \quad (1)$$

A special solution, the only separable solution⁷, to axioms (A1)-(A4) is the Ross solution: $p_{ij} = \bar{p}_j$ and $q_{ji} = \bar{q}_i$, where

$$\bar{p}_j = \frac{b_j T_j^f}{\sum_{i=1}^L c_i T_i^m} \quad \text{and} \quad \bar{q}_i = \frac{c_i T_i^m}{\sum_{j=1}^N b_j T_j^f} \quad (2)$$

(both are implicit functions of time).

Castillo-Chavez and Busenberg⁹ characterized all solutions to axioms (A1)-(A4) as multiplicative perturbations of the Ross solution. The perturbations were defined in terms of two matrices, $\Phi^m = \{\phi_{ij}^m\}$ and $\Phi^f = \{\phi_{ji}^f\}$. The matrices Φ^m and Φ^f define the preferences and/or affinities of types of individuals of one gender for other types (here of the opposite gender), and these preferences may change with time directly or through changes in the frequency of the types. We refer to these two matrices as the male and female *preference matrices*, respectively. To present some new results, we need to explicitly state our characterization theorem of two-sex marriage functions. The following expressions are needed:

$$\begin{aligned} \ell_i^m &\equiv \sum_{j=1}^N \bar{p}_j \phi_{ij}^m, & R_i^m &\equiv 1 - \ell_i^m, & V^m &\equiv \sum_{i=1}^L \bar{q}_i R_i^m, \\ \ell_j^f &\equiv \sum_{i=1}^L \bar{q}_i \phi_{ji}^f, & R_j^f &\equiv 1 - \ell_j^f, & V^f &\equiv \sum_{j=1}^N \bar{p}_j R_j^f. \end{aligned} \quad (3)$$

Theorem 1.⁹

For each marriage function (P, Q) , matrices Φ^m and Φ^f can be found so that

$$p_{ij} = \bar{p}_j \left[\frac{R_j^f R_i^m}{V^f} + \phi_{ij}^m \right] \quad \text{and} \quad q_{ji} = \bar{q}_i \left[\frac{R_i^m R_j^f}{V^m} + \phi_{ji}^f \right] \quad (4)$$

with $0 \leq R_i^m \leq 1$ and $0 \leq R_j^f \leq 1$ for $i=1, \dots, L$, $j=1, \dots, N$, and $\sum_{i=1}^L \ell_i^m \bar{q}_i < 1$ and $\sum_{j=1}^N \ell_j^f \bar{p}_j < 1$ if and only if

$$\phi_{ij}^m = \phi_{ji}^f + R_i^m R_j^f \left[\frac{1}{V^m} - \frac{1}{V^f} \right]. \quad (5)$$

The conditions $0 \leq R_i^m \leq 1$ and $0 \leq R_j^f \leq 1$ imply that $0 \leq \ell_i^m \leq 1$ and $0 \leq \ell_j^f \leq 1$, while the inequalities $\sum_{i=1}^L \ell_i^m \bar{q}_i < 1$ and $\sum_{j=1}^N \ell_j^f \bar{p}_j < 1$ guarantee that $V^m > 0$ and $V^f > 0$. Expression (4) reveals that p_{ij} and q_{ji} are implicit functions of frequencies (and time). Condition (5) shows the implicit frequency (and time) dependent relationship forced by (A3) between the elements of Φ^m and Φ^f . Let

$$\vec{\bar{p}} = \begin{pmatrix} \bar{p}_1 \\ \vdots \\ \bar{p}_N \end{pmatrix} \quad \text{and} \quad \vec{\bar{q}} = \begin{pmatrix} \bar{q}_1 \\ \vdots \\ \bar{q}_L \end{pmatrix}.$$

Using matrix notation, we can combine the constraints imposed by (5) in an implicit nonlinear relationship of the following general form

$$\Phi^m = \psi(\vec{\bar{p}}, \vec{\bar{q}}, \Phi^f, \Phi^m), \quad (6)$$

where the elements of ψ are defined component-wise by (5). The nonlinear expression (6) succinctly summarizes the constraints imposed by (A3) on the mixing subpopulations and their defining parameters.

We conclude this section with a useful result which gives an insight into the role of Φ^m and Φ^f :

Theorem 2.⁹

If either $\phi_{ij}^m = \alpha$, $0 \leq \alpha < 1 \quad \forall i, j$, or $\phi_{ji}^f = \beta$, $0 \leq \beta < 1 \quad \forall j, i$, where α and β are constants, then $p_{ij} = \bar{p}_j$ and $q_{ji} = \bar{q}_i$. That is, Equation (4) reduces to the unique separable Ross solution in (2).

3. Parametrization of Preference Matrices

Equation (4) encapsulates all possible mixing patterns in terms of two preference matrices. It

may be argued that this representation just passes the buck by transferring the difficulties from one set of matrices, (P, Q) , to another, (Φ^m, Φ^f) . In fact, mixing between individuals is a complex process which is not really possible to get around. The use of preference matrices (Φ^m, Φ^f) helps increase our understanding of the marriage/social structure of a population. Preference matrices facilitate the modeling of specific, non-trivial mixing patterns between individuals. Earlier theoretical work was based on random, or specific types of assortative mating, and few other variations, particularly in population genetics¹⁸. Modelers, who were interested in mating systems at the level of the individual, began to move away from random mating through the use of special mixing matrices including like-with-like, preferred mixing, or biased mixing¹⁹. Other forms of mixing such as those females preferred to mix with older males and males preferred to mixed with younger females were avoided because either they led to intractable mathematical models or there was no obvious way of modeling this type of mixing. The consequences of this type of self-imposed limitation have just begun to be explored. For example, disease-dynamics and demographic studies were based on models constructed with unrealistic mating structures. The question that must be asked is, to what extent are these results too dependent on the used of specialized forms of mixing?

Preference matrices (Φ^m, Φ^f) help construct more realistic mating/social structures. In this section, we present a simple result that allows for the modeling of flexible mixing structures parametrized with very few parameters. We illustrate the use of these matrices with real data, that is, with mixing matrices that we constructed using the method that we published in this journal¹⁷. The key result is expressed in the following theorem:

Theorem 3. $V^f = V^m$ if and only if $\Phi^m = (\Phi^f)^T$, where τ denotes transposition.

Proof. It is immediate from Equation (5) that $V^f = V^m$ implies that $\phi_{ij}^m = \phi_{ji}^f$.

Since

$$\begin{aligned} V^f &= \sum_{j=1}^N \bar{p}_j R_j^f = \sum_{j=1}^N \bar{p}_j [1 - \ell_j^f] = 1 - \sum_{j=1}^N \bar{p}_j \sum_{i=1}^L \bar{q}_i \phi_{ji}^f \\ &= 1 - \sum_{j=1}^N \sum_{i=1}^L \bar{p}_j \bar{q}_i \phi_{ji}^f, \end{aligned} \tag{7}$$

consequently, if $\phi_{ij}^m = \phi_{ji}^f \ \forall i, j$, then

$$\begin{aligned} V^f &= 1 - \sum_{j=1}^N \sum_{i=1}^L \bar{p}_j \bar{q}_i \phi_{ij}^m = 1 - \sum_{i=1}^L \bar{q}_i \sum_{j=1}^N \bar{p}_j \phi_{ij}^m \\ &= \sum_{i=1}^L \bar{q}_i [1 - \ell_i^m] = \sum_{i=1}^L \bar{q}_i R_i^m = V^m. \end{aligned} \quad \square$$

The above result implies that the only solutions to axioms (A1)-(A4) with frequency independent Φ^m and Φ^f are those with $\Phi^m = (\Phi^f)^T$. Namely, males and females have matching preferences which do not change with $T_i^m(t)$ and $T_j^f(t)$. Although the class of solutions with $\Phi^m = (\Phi^f)^T$ is quite restrictive, this class extends, considerably, the mixing/mating structures available in the literature. Furthermore, if we use constant preference matrices Φ^m and Φ^f , then the class of parametric mixing models becomes quite rich and flexible. Figure 1 shows a real mixing matrix, which is also listed in Table 2. It was constructed using our data from a known population of undergraduate students and their partners²⁰. The data summary is listed in Table 1. The known (targeted) population was stratified by school year as class 1 for freshmen, class 2 for sophomores, class 3 for juniors and class 4 for seniors, while those partners who were not members of the known population were assigned to an additional class of their own, the non-targeted population, or class 5. This stratification is highly correlated with age and, consequently, we can also “read” the age-structure mixing from this matrix. Figure 1 shows strong evidence of like-with-like mixing (individuals prefer to mix with those of the same class or age) coupled with an additional trend, namely, females tend to pair with older males and males tend to pair with younger females. In addition, it shows that the link between targeted (classes 1, 2, 3 and 4) and non-targeted (class 5) populations is very strong. Thus, the use of constant preference matrices that satisfy the relationship $\Phi^m = (\Phi^f)^T$ provides a reasonable first approximation for the construction of a mixing parametric model.

Table 1 about here

Table 2 about here

Figure 1 about here

Figure 2 about here

The model most commonly used in the past is that of random mixing (the Ross solution). Figure 2 shows the corresponding random mixing pattern associated with our data. Clearly the Ross solution does not capture the features observed in the data. We propose Φ^f matrices of one of the following types

$$\Phi_1^f = \begin{bmatrix} 1 & d & d & d & d \\ 0 & 1 & d & d & d \\ 0 & 0 & 1 & d & d \\ 0 & 0 & 0 & 1 & d \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{or} \quad \Phi_2^f = \begin{bmatrix} d & d & d & d & d \\ 0 & d & d & d & d \\ 0 & 0 & d & d & d \\ 0 & 0 & 0 & d & d \\ 0 & 0 & 0 & 0 & d \end{bmatrix}, \quad (8)$$

and hope to capture the qualitative feature observed in the data with a single parameter, d , under the assumption that $\Phi^m = (\Phi^f)^T$. Other types of Φ^f might fit our data as well. The following example, shows that mixing matrices, parametrized in the above fashion, provide useful and realistic parametric models.

For our illustrative example we assume that data were collected at $t = 0$ and that the pair-formation rates and initial population sizes $\{c_i, b_j, T_i^m(0), T_j^f(0) : i=1, \dots, L; j=1, \dots, N\}$ are known (Table 1). The mixing matrix $(\hat{P}(0), \hat{Q}(0))$ is estimated using our recently developed methods¹⁷ and our own data¹⁵, which is presented in Figure 1 and Table 2. We estimate the parameter d in models Φ_1^f and Φ_2^f using least squares. For a given d , the predicted mixing matrix, $(\tilde{P}(0), \tilde{Q}(0))_d$, is given by (4) (or by (34) in the next section). The least squares estimate of d , \hat{d} , minimizes the sum of squared differences of all corresponding elements between $(\hat{P}(0), \hat{Q}(0))$ and $(\tilde{P}(0), \tilde{Q}(0))_d$ given that the range of d is constrained by the conditions $0 \leq \ell_i^m \leq 1$ and $0 \leq \ell_j^f \leq 1$. If $\Phi^f = \Phi_1^f$, then the lower bound of d is nonpositive, and the upper bound is 1 because

$$1 = \frac{1 - \bar{q}_1}{\sum_{i=2}^L \bar{q}_i} \leq \frac{1 - \bar{q}_2}{\sum_{i=3}^L \bar{q}_i} \leq \dots \leq \frac{1 - \bar{q}_{L-1}}{\bar{q}_L}$$

and

$$1 = \frac{1 - \bar{p}_N}{\sum_{j=1}^{N-1} \bar{p}_j} \leq \frac{1 - \bar{p}_{N-1}}{\sum_{j=1}^{N-2} \bar{p}_j} \leq \dots \leq \frac{1 - \bar{p}_2}{\bar{p}_1}.$$

If $\Phi^f = \Phi_2^f$, then the constraint reduces to $0 \leq d \leq 1$ because

$$1 = \frac{1}{\sum_{i=1}^L \bar{q}_i} \leq \frac{1}{\sum_{i=2}^L \bar{q}_i} \leq \dots \leq \frac{1}{\bar{q}_L}$$

and

$$1 = \frac{1}{\sum_{j=1}^N \bar{p}_j} \leq \frac{1}{\sum_{j=1}^{N-1} \bar{p}_j} \leq \dots \leq \frac{1}{\bar{p}_1}.$$

Because our data come from a non-closed network, we¹⁷ developed a methodology based on mark-recapture techniques that allows us to conditionally estimate the size of the non-targeted population. Furthermore, with the use of properties (A1)-(A3) we¹³⁻¹⁵ were able to estimate $(P(0), Q(0))$ using the five classes as shown in Figure 1 and Table 2. Along with the estimated population sizes and pair-formation rates in Table 1, we compute the least squares estimate of d for Φ_1^f and Φ_2^f . The estimates are: $\hat{d}_1 \approx 0.50$ for $\Phi^f = \Phi_1^f$, and $\hat{d}_2 \approx 0.48$ for $\Phi^f = \Phi_2^f$. The predicted mixing matrices with \hat{d} , $(\tilde{P}, \tilde{Q})_{\hat{d}}$, are presented in Figures 3 and 4.

Figure 3 about here

Figure 4 about here

Since the sum of squared errors with \hat{d}_1 is smaller (0.699) than that with \hat{d}_2 (0.797), the model with $\Phi^f = \Phi_1^f$ is preferable to that with $\Phi^f = \Phi_2^f$. It is also clear if one compares Figures 3 and 4 with Figure 1.

If the preference matrix $\Phi^m = (\Phi^f)^T$ is fixed, that is, if all their elements are constant, then relationship (5) is always satisfied. Once we have computed \hat{d} from the data, we can predict p_{ij} and q_{ji} for all times by (4) or (34) through the dynamics of the population, *i.e.*, $T_i^m(t)$ and $T_j^f(t)$, which can be predicted using deterministic or stochastic models. In fact, we¹¹ have constructed stochastic and deterministic demographic and epidemiological models that incorporate the contact structure described in (4).

4. The Two Body Problem or it Takes Two to Tango

In Section 3, we found the relationship

$$\Phi^m = \psi\left(\vec{p}, \vec{q}, \Phi^f, \Phi^m\right),$$

or, in other words, the preferences of males for females and viceversa satisfy a complex relationship unless they both “agree”, that is, $\Phi^m = (\Phi^f)^T$. Common sense dictates that if the set of preference of one gender (e.g., Φ^f) is known then so must be the other (e.g., Φ^m). Consequently, we should be able to solve (5), that is, we should be able to compute a relationship like

$$\Phi^m = \Psi\left(\vec{p}, \vec{q}, \Phi^f\right).$$

Hence, Φ^m is just a function of the Ross solution and Φ^f , and a simpler relationship is not possible in general. This result will be referred as the “T³ Theorem” as it convincingly shows that, in all situations, it “Takes Two to Tango”.

To find a solution of Φ^m in terms of \bar{p}_j , \bar{q}_i and ϕ_{ji}^f , we multiply \bar{p}_j on both sides of Equation (5) and sum over j . The resulting relationships are:

$$\begin{aligned} \sum_{j=1}^N \bar{p}_j \phi_{ij}^m &= \sum_{j=1}^N \bar{p}_j \phi_{ji}^f + \sum_{j=1}^N \bar{p}_j R_i^m R_j^f \left[\frac{1}{V^m} - \frac{1}{V^f} \right] \\ \Leftrightarrow 1 - R_i^m &= \sum_{j=1}^N \bar{p}_j \phi_{ji}^f + R_i^m V^f \left[\frac{1}{V^m} - \frac{1}{V^f} \right] \\ \Leftrightarrow 1 - \sum_{j=1}^N \bar{p}_j \phi_{ji}^f &= R_i^m \left[\frac{V^f}{V^m} \right]. \end{aligned} \tag{9}$$

If we define

$$U_i^f \equiv \sum_{j=1}^N \bar{p}_j \phi_{ji}^f, \tag{10}$$

then from Equation (9) we obtain

$$\frac{1 - U_i^f}{V^f} = \frac{R_i^m}{V^m}, \tag{11}$$

which reveals the fact that male preferences can be obtained from female preference if we can solve (11). Explicitly, if we define

$$\beta_i^f \equiv \frac{1 - U_i^f}{V^f}, \tag{12}$$

then the system that must be solved becomes

$$R_i^m - \beta_i^f V^m = 0, \tag{13}$$

or

$$R_i^m - \beta_i^f \sum_{k=1}^L \bar{q}_k R_k^m = 0 , \quad (14)$$

or in matrix notation

$$\left(\mathbf{I} - \vec{\beta}^f \vec{\bar{q}}^T \right) \vec{R}^m = \vec{0} , \quad (15)$$

where

$$\vec{\beta}^f = \begin{pmatrix} \beta_1^f \\ \vdots \\ \beta_L^f \end{pmatrix} \quad \text{and} \quad \vec{R}^m = \begin{pmatrix} R_1^m \\ \vdots \\ R_L^m \end{pmatrix} .$$

If we let

$$\mathbf{B} = \mathbf{I} - \vec{\beta}^f \vec{\bar{q}}^T , \quad (16)$$

then we observe that \mathbf{B} , an $L \times L$ matrix, is rank one perturbation of the identity. Furthermore,

$$\det \mathbf{B} = 1 - \sum_{i=1}^L \bar{q}_i \beta_i^f = 0 , \quad (17)$$

which can be seen from the following simple calculations:

$$\det \mathbf{B} = 1 - \frac{\sum_{i=1}^L \bar{q}_i (1 - U_i^f)}{V^f} = 1 - \frac{1 - \sum_{i=1}^L \bar{q}_i U_i^f}{V^f} \quad (18)$$

$$= 1 - \frac{1 - \sum_{i=1}^L \bar{q}_i \sum_{j=1}^N \bar{p}_j \phi_{ji}^f}{V^f} = 1 - \frac{1 - \sum_{j=1}^N \sum_{i=1}^L \bar{p}_j \bar{q}_i \phi_{ji}^f}{V^f} \quad (19)$$

$$= 1 - \frac{V^f}{V^f} = 0 \quad (\text{using (7)}).$$

Note that Equations (18) and (19) also imply

$$V^f = 1 - \sum_{i=1}^L \bar{q}_i U_i^f . \quad (20)$$

Since $\det \mathbf{B} = 0$, all solutions are given by

$$\vec{R}^m = \gamma \vec{\beta}^f , \quad (21)$$

where γ is an arbitrary “constant” for each time t . In other words, the null space of $\mathbf{I} - \vec{\beta}^f \vec{\bar{q}}^T$ is

equal to $\text{span}\{\vec{\beta}^f\}$. To verify this result, we substitute the solution in (21) into (14) and obtain

$$\text{LHS} = \gamma \beta_i^f - \beta_i^f \sum_{k=1}^L \bar{q}_k \gamma \beta_k^f = \gamma \beta_i^f \left(1 - \sum_{k=1}^L \bar{q}_k \beta_k^f\right) = 0 = \text{RHS}$$

for all i , where $\left(1 - \sum_{k=1}^L \bar{q}_k \beta_k^f\right) = \det \mathbf{B} = 0$.

Substituting the solution (21) into (5), we obtain

$$\phi_{ij}^m = \phi_{ji}^f + \gamma \beta_i^f R_j^f \left[\frac{1}{\gamma} - \frac{1}{V^f}\right] = \phi_{ji}^f + \beta_i^f R_j^f \left[1 - \frac{\gamma}{V^f}\right]. \quad (22)$$

From the condition $\sum_{j=1}^N \ell_j^f \bar{p}_j < 1$ in Theorem 1, $V^f > 0$ for all times because

$$\sum_{j=1}^N \ell_j^f \bar{p}_j = \sum_{j=1}^N \sum_{i=1}^L \bar{q}_i \phi_{ji}^f \bar{p}_j = \sum_{i=1}^L \bar{q}_i U_i^f < 1, \quad (23)$$

and thus by Equation (20)

$$V^f = 1 - \sum_{i=1}^L \bar{q}_i U_i^f > 0.$$

If we further constrain ϕ_{ji}^f to assure that $U_i^f = \sum_{j=1}^N \bar{p}_j \phi_{ji}^f \leq 1$, then $\beta_i^f \geq 0$ for all i by (12) as negative values of β_i^f would imply that $U_i^f > 1$. Finally, in order to have $0 \leq R_i^m \leq 1$, we need to choose γ satisfying

$$0 \leq \gamma \leq \frac{1}{\max_i \beta_i^f}. \quad (24)$$

Note that not all β_i^f can be zero, or not all U_i^f can be one, otherwise $V^f = 0$. The parameter γ gives an extra degree of freedom in the choice of R_i^m and ϕ_{ij}^m . To reparametrize or rescale the free parameter, we let

$$\Gamma = 1 - \frac{\gamma}{V^f}, \quad (25)$$

or

$$\gamma = V^f (1 - \Gamma). \quad (26)$$

Hence, Equation (22) becomes

$$\phi_{ij}^m = \phi_{ji}^f + \beta_i^f R_j^f \left[1 - \frac{V^f (1 - \Gamma)}{V^f}\right] = \phi_{ji}^f + \Gamma \beta_i^f R_j^f. \quad (27)$$

Plugging Equation (26) into Inequality (24) results in

$$\begin{aligned}
 0 &\leq V^f (1 - \Gamma) \leq \frac{1}{\max_i \beta_i^f} \\
 \Leftrightarrow \quad 0 &\leq 1 - \Gamma \leq \frac{1}{V^f \max_i \beta_i^f} \\
 \Leftrightarrow \quad 1 - \frac{1}{V^f \max_i \beta_i^f} &\leq \Gamma \leq 1 ,
 \end{aligned}$$

which by Equation (12) holds

$$\begin{aligned}
 \Leftrightarrow \quad 1 - \frac{1}{1 - \min_i U_i^f} &\leq \Gamma \leq 1 \\
 \Leftrightarrow \quad \frac{-\min_i U_i^f}{1 - \min_i U_i^f} &\leq \Gamma \leq 1 .
 \end{aligned} \tag{28}$$

These preliminary computations allow us to state our main result, the T^3 Theorem:

T^3 Theorem. The preference matrices at all times obey the following explicit relation:

$$\phi_{ij}^m = \phi_{ji}^f + \Gamma \frac{(1 - U_i^f) R_j^f}{V^f} , \tag{29}$$

where Γ is an implicitly time-dependent arbitrary “constant” satisfying

$$\frac{-\min_i U_i^f}{1 - \min_i U_i^f} \leq \Gamma \leq 1 ;$$

and conversely

$$\phi_{ji}^f = \phi_{ij}^m + \Delta \frac{(1 - U_j^m) R_i^m}{V^m} , \tag{30}$$

where

$$U_j^m = \sum_{i=1}^L \bar{q}_i \phi_{ij}^m , \tag{31}$$

and Δ is an implicitly time-dependent arbitrary “constant” satisfying

$$\frac{-\min_j U_j^m}{1 - \min_j U_j^m} \leq \Delta \leq 1 . \tag{32}$$

The function Ψ is thus defined explicitly by Equation (29).

If $\phi_{ji}^f = \alpha$ (constant, and $0 \leq \alpha < 1$) for all i, j , then $0 < R_j^f = 1 - \alpha \leq 1$ for all j by (3), $0 \leq U_i^f = \alpha < 1$ for all i by (10), and $0 < V^f = 1 - \alpha \leq 1$ by (20). Thus, $\beta_i^f = \frac{1-\alpha}{1-\alpha} = 1$ by (12) and $0 \leq R_i^m = \gamma \leq 1$ by (21) for all i . Hence, from (22)

$$\phi_{ij}^m = \alpha + (1 - \alpha) \left[1 - \frac{\gamma}{1 - \alpha} \right] = 1 - \gamma = \text{constant}$$

for all i, j , and $0 \leq 1 - \gamma < 1$. This is an alternate proof of Theorem 2 of Section 3, which states that lack of selectivity (preference) in one sex implies lack of selectivity (preference) in the other. In this case, $p_{ij} = \bar{p}_j$ and $q_{ji} = \bar{q}_i$, that is, the population mixes at random. If $\Gamma = 0$ or $\Delta = 0$ for all times, then $\Phi^m = (\Phi^f)^T$, the frequency independent mixing matrices of Theorem 3.

If the female preferences dominate and are hypothesized as $\tilde{\phi}_{ji}^f$, and the actual male preferences are given by $\bar{\phi}_{ij}^m$, then we can obtain the optimal hypothesized male preferences $\tilde{\phi}_{ij}^m$ through (27) or (29) by choosing Γ so as to minimize the sum of squared differences between $\bar{\phi}_{ij}^m$ and $\tilde{\phi}_{ij}^m$. Thus, the choice of Γ is given by

$$\Gamma = \frac{\sum_{i=1}^L \sum_{j=1}^N (\bar{\phi}_{ij}^m - \tilde{\phi}_{ij}^m) \beta_i^f R_j^f}{\sum_{i=1}^L \sum_{j=1}^N (\beta_i^f R_j^f)^2}. \quad (33)$$

A similar result holds through (30) if the male preferences dominate and the actual female preferences are given. In addition, one can also derive expressions for Γ and Δ if one assigns a weighted preference dominance to each of the two sexes.

Using the mixing solution given by (21) and (22), the general mixing matrix in (4) can be rewritten as follows:

$$\begin{aligned} p_{ij} &= \bar{p}_j \left[\frac{R_j^f \gamma \beta_i^f}{V^f} + \phi_{ji}^f + \beta_i^f R_j^f \left(1 - \frac{\gamma}{V^f} \right) \right] = \bar{p}_j \left[\beta_i^f R_j^f + \phi_{ji}^f \right], \\ q_{ji} &= \bar{q}_i \left[\frac{\gamma \beta_i^f R_j^f}{\gamma} + \phi_{ji}^f \right] = \bar{q}_i \left[\beta_i^f R_j^f + \phi_{ji}^f \right]. \end{aligned} \quad (34)$$

The terms containing γ are canceled. If we visualize the preference matrices Φ^m and Φ^f as behavioral “genotypes”, then (34) which expresses the mating system as a function of the behavioral “genotype”

turns out to be independent of γ . Therefore, behavioral “genotypes” with different γ can give rise to identical behavioral “phenotypes”, that is, to the same set of mixing probabilities.

5. Conclusions

The word “marriage” tacitly implies the involvement of two individuals and, consequently, the possibility that the behavior of “single” individuals will influence the behavior/decision of his/her potential partners. Thus, two-sex marriage functions must indeed be complicated and therefore, random mating marriage functions have strong limitations. The axiomatic framework developed by Busenberg and Castillo-Chavez⁷⁻⁹ provides a systematic approach to the study of marriage functions: all marriage functions are characterized as multiplicative perturbations of the Ross solution via a pair of preference matrices. These preference matrices are intimately connected and knowledge of one implies knowledge of the other. The general relationship between the two preference matrices is stated as the T^3 Theorem and involves a free parameter. The range of this parameter is constrained by explicitly stated conditions. However, this free parameter does not contribute to the calculation of the marriage functions. In other words, the value of the free parameter models a class of behavioral “genotypes” that give rise to identical behavioral “phenotypes”.

The only frequency-independent preference matrices are those satisfy $\Phi^m = (\Phi^f)^T$, which are supported by our data. We propose two types of Φ^f with one parameter which can be estimated from data using least-squares criterion. Other types of Φ^f are being explored.

The application of marriage functions is of importance in various fields such as cultural anthropology, demography, ecology and evolutionary biology, social dynamics, and epidemiology. The exploration of general mating functions in specific biological or sociological settings is yet to be explored. However, our^{10,11} initial results are quite promising and alternative approaches have been utilized by others^{21,22}. In this paper, we use fixed classification for the stratification of individuals, extensions to populations stratified by dynamic classifications such as age or age of infection are being worked out^{7,8,16}.

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Table 1. Population sizes and pair-formation rates estimated from data

Class	Male		Female	
	Population Size	Pair-Formation Rate	Population Size	Pair-Formation Rate
i/j	$T_i^m(0)$	c_i	$T_j^f(0)$	b_j
1	339	1.69	376	1.45
2	636	1.33	500	1.38
3	505	1.30	754	1.17
4	897	1.40	674	1.04
5	2343	1.50	1796	2.24

Table 2. Male, $\hat{P}(0)$, and female, $\hat{Q}(0)$, mixing matrices estimated from data

Subject Class		Partner Class j/i				
	i/j	1	2	3	4	5
$\hat{p}_{ij}(0)$	1	0.444	0.111	0.037	0.037	0.370
	2	0.063	0.281	0.063	0.063	0.531
	3	0.000	0.154	0.154	0.154	0.538
	4	0.029	0.114	0.200	0.314	0.343
	5	0.057	0.041	0.129	0.036	0.736
$\hat{q}_{ji}(0)$	1	0.172	0.103	0.138	0.103	0.483
	2	0.028	0.361	0.139	0.056	0.417
	3	0.048	0.095	0.262	0.167	0.429
	4	0.000	0.000	0.034	0.241	0.724
	5	0.103	0.114	0.057	0.210	0.516

Figure 1. Mixing matrix estimated from data.

- (a1) Male, $\hat{P}(0)$, 3D plot; (a2) Male, $\hat{P}(0)$, contour plot;
(b1) Female, $\hat{Q}(0)$, 3D plot; (b2) Female, $\hat{Q}(0)$, contour plot.

Figure 2. Random mixing matrix for our data.

- (a1) Male, 3D plot; (a2) Male, contour plot;
(b1) Female, 3D plot; (b2) Female, contour plot.

Figure 3. Mixing matrix for our data with preference matrices $\Phi^m = (\Phi_1^f)^T$ and $\hat{d} = 0.5$.

- (a1) Male, $\tilde{P}(0)_{0.5}$, 3D plot; (a2) Male, $\tilde{P}(0)_{0.5}$, contour plot;
(b1) Female, $\tilde{Q}(0)_{0.5}$, 3D plot; (b2) Female, $\tilde{Q}(0)_{0.5}$, contour plot.

Figure 4. Mixing matrix for our data with preference matrices $\Phi^m = (\Phi_2^f)^T$ and $\hat{d} = 0.48$.

- (a1) Male, $\tilde{P}(0)_{0.48}$, 3D plot; (a2) Male, $\tilde{P}(0)_{0.48}$, contour plot;
(b1) Female, $\tilde{Q}(0)_{0.48}$, 3D plot; (b2) Female, $\tilde{Q}(0)_{0.48}$, contour plot.

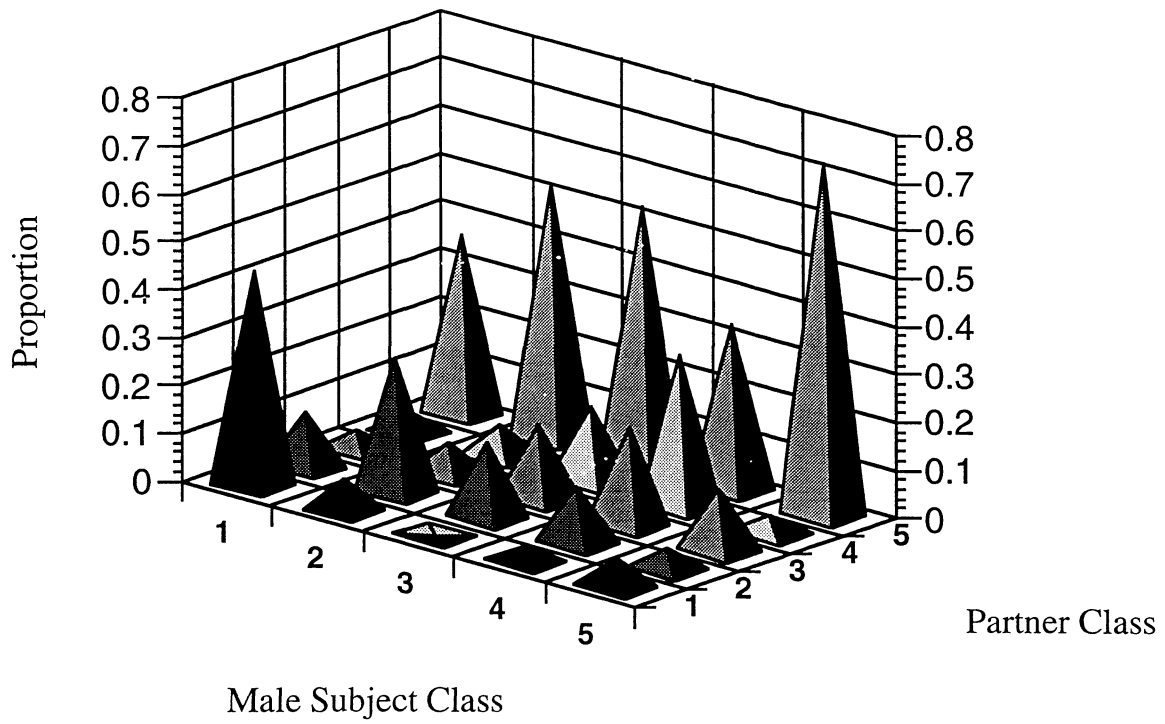


Figure 1 (a1)

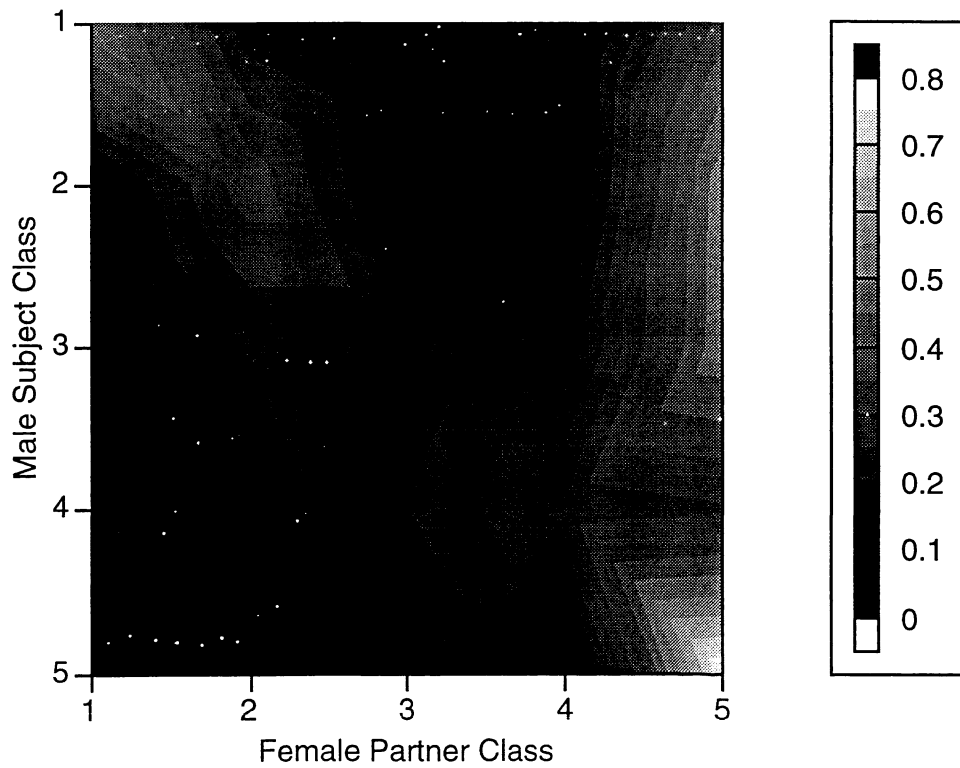


Figure 1 (a2)

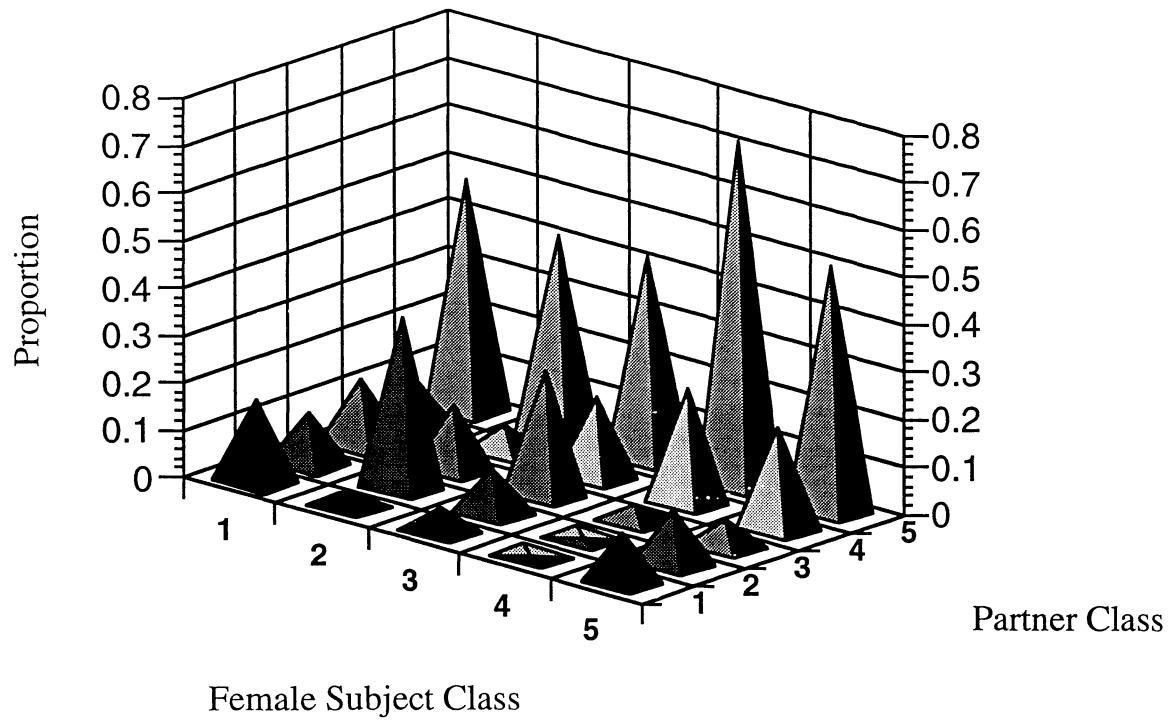


Figure 1 (b1)

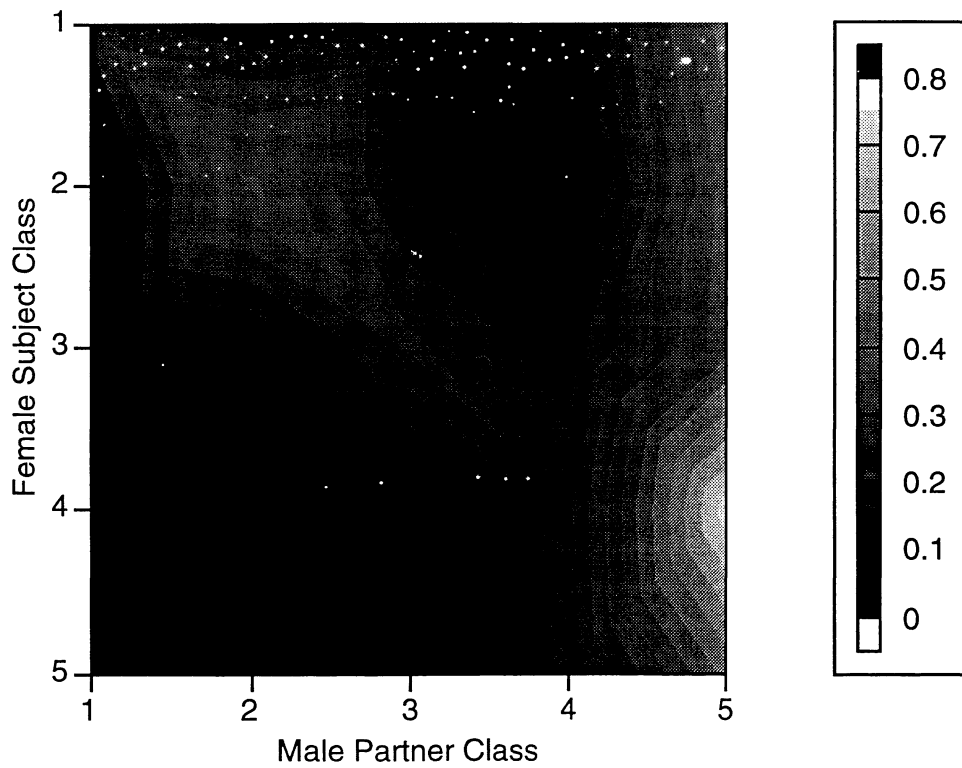


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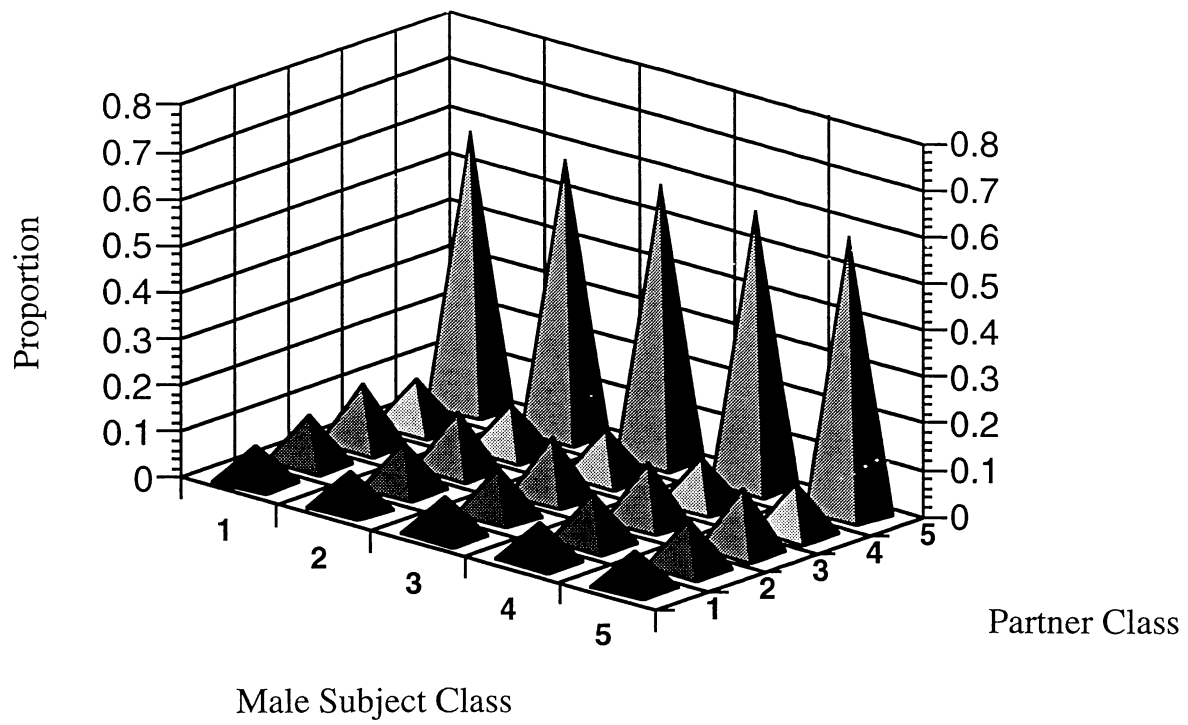


Figure 2 (a1)

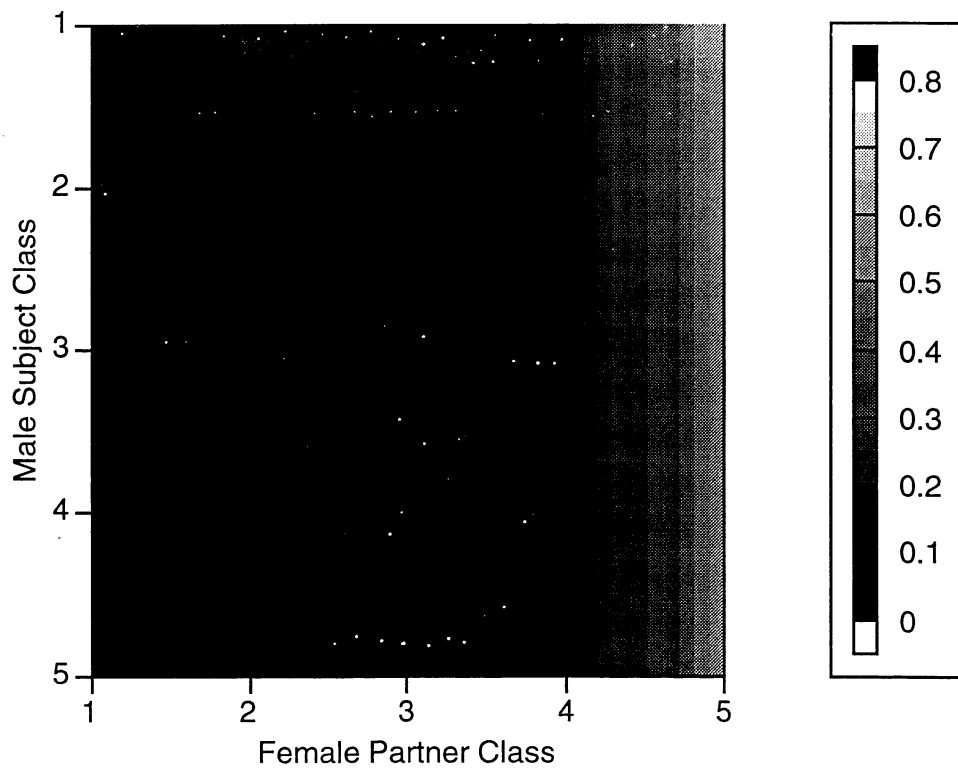


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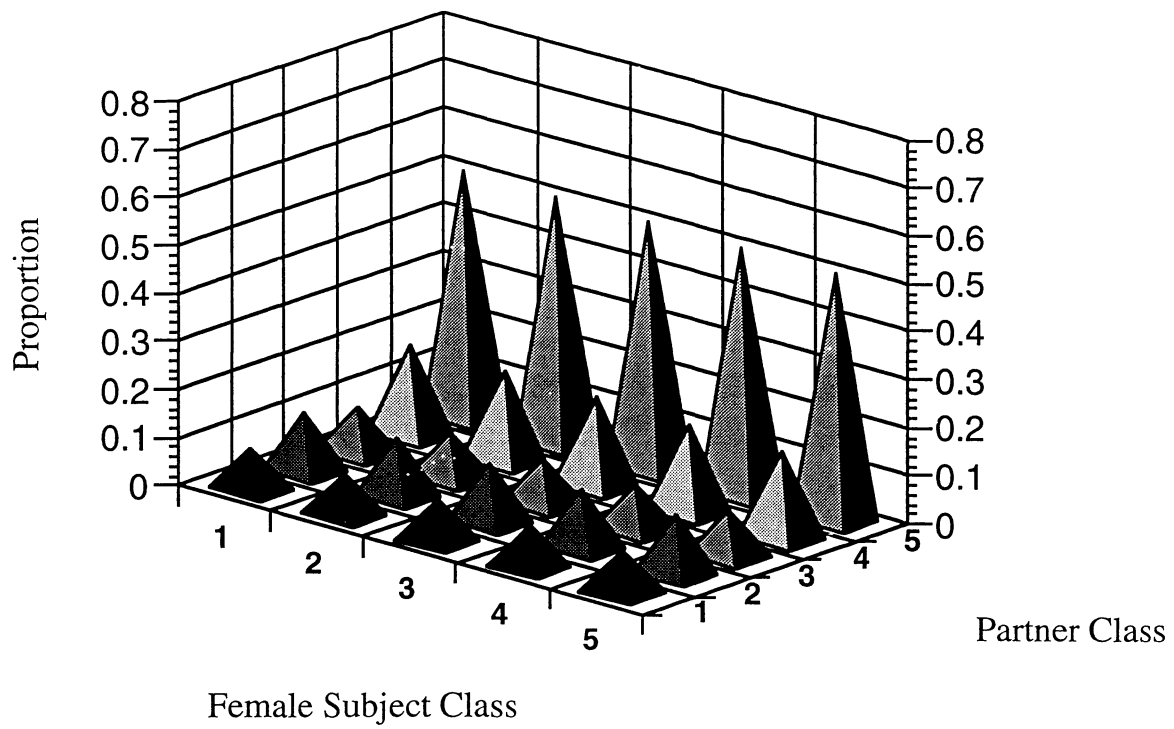


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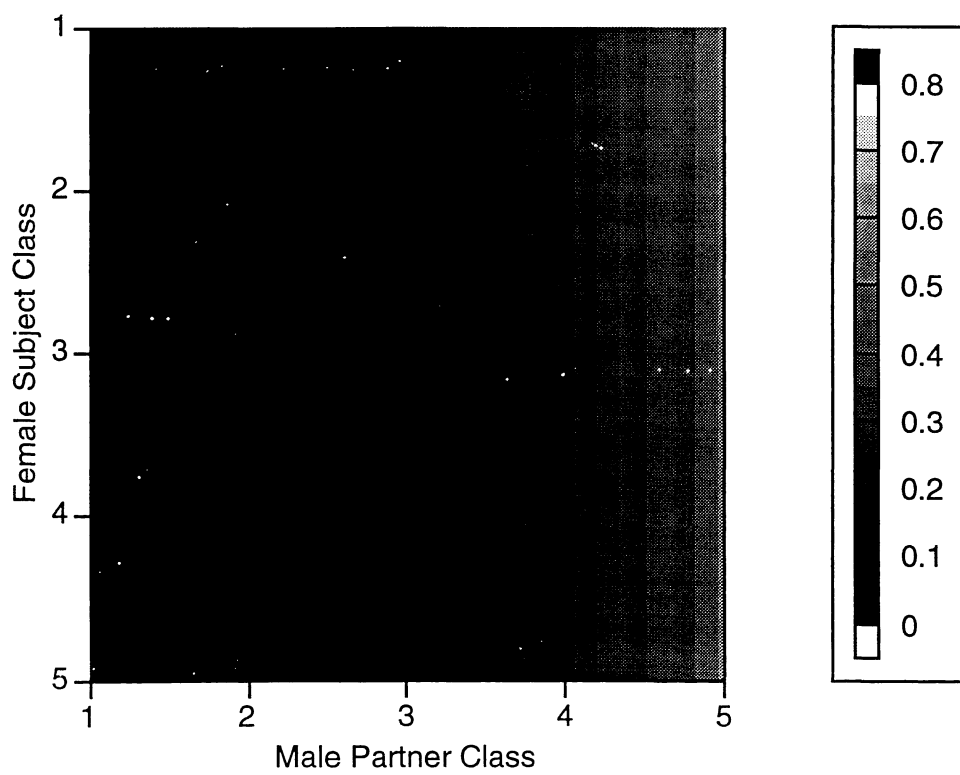


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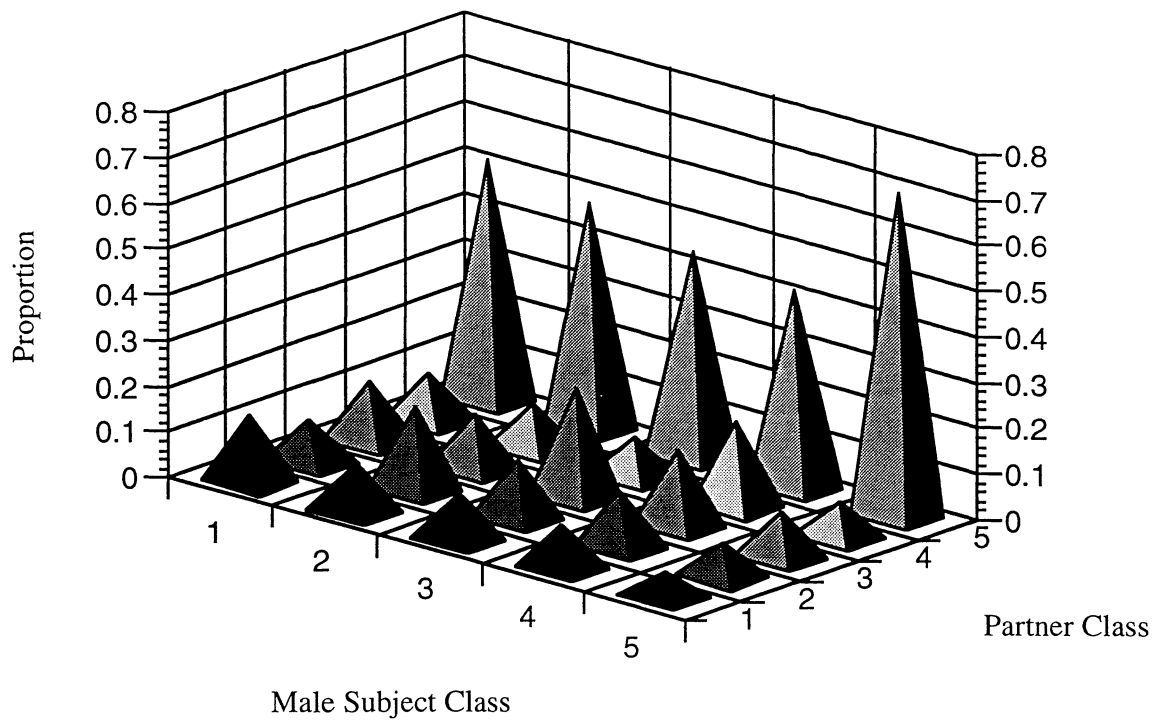


Figure 3 (a1)

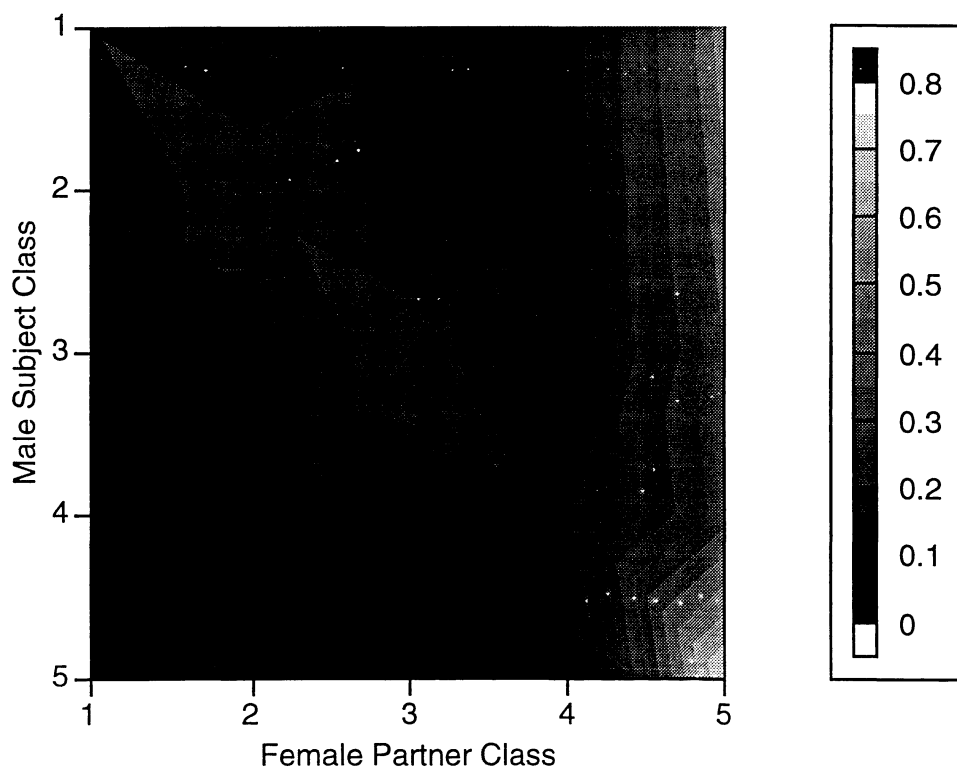


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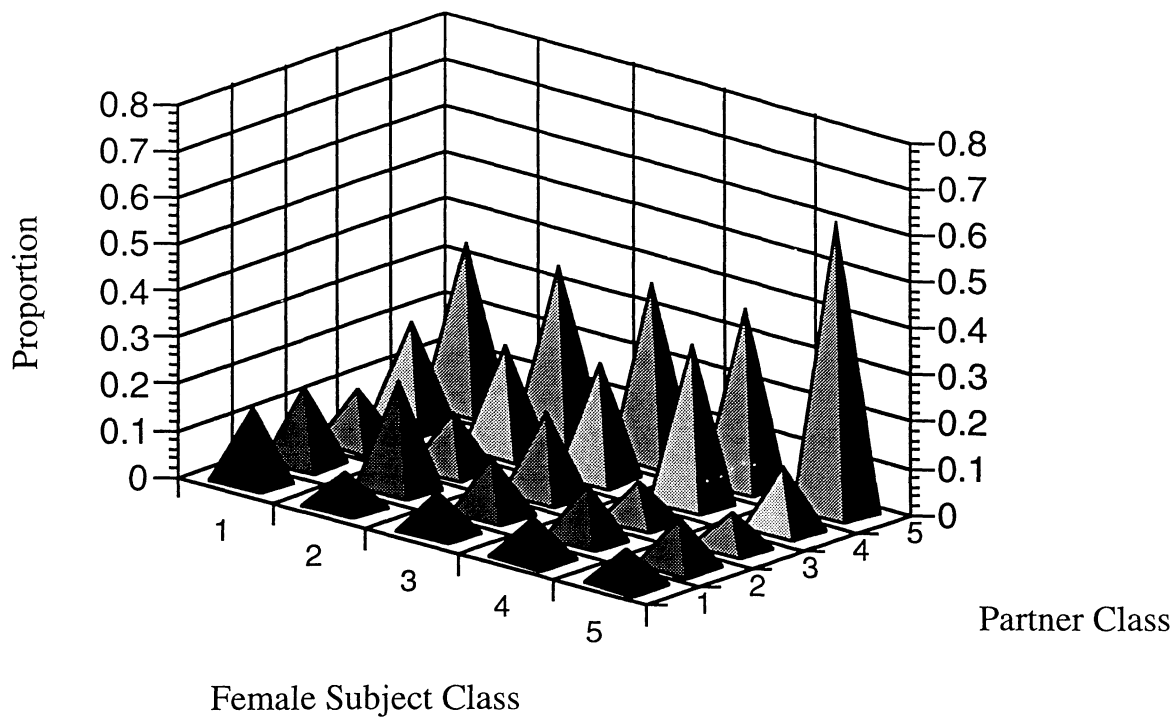


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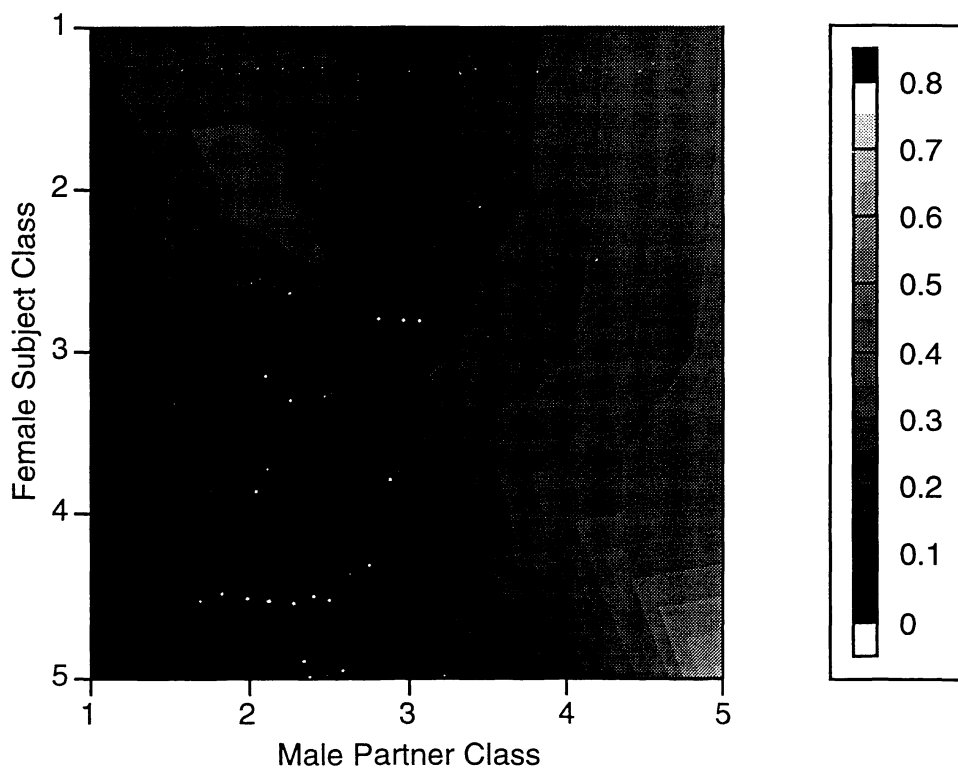


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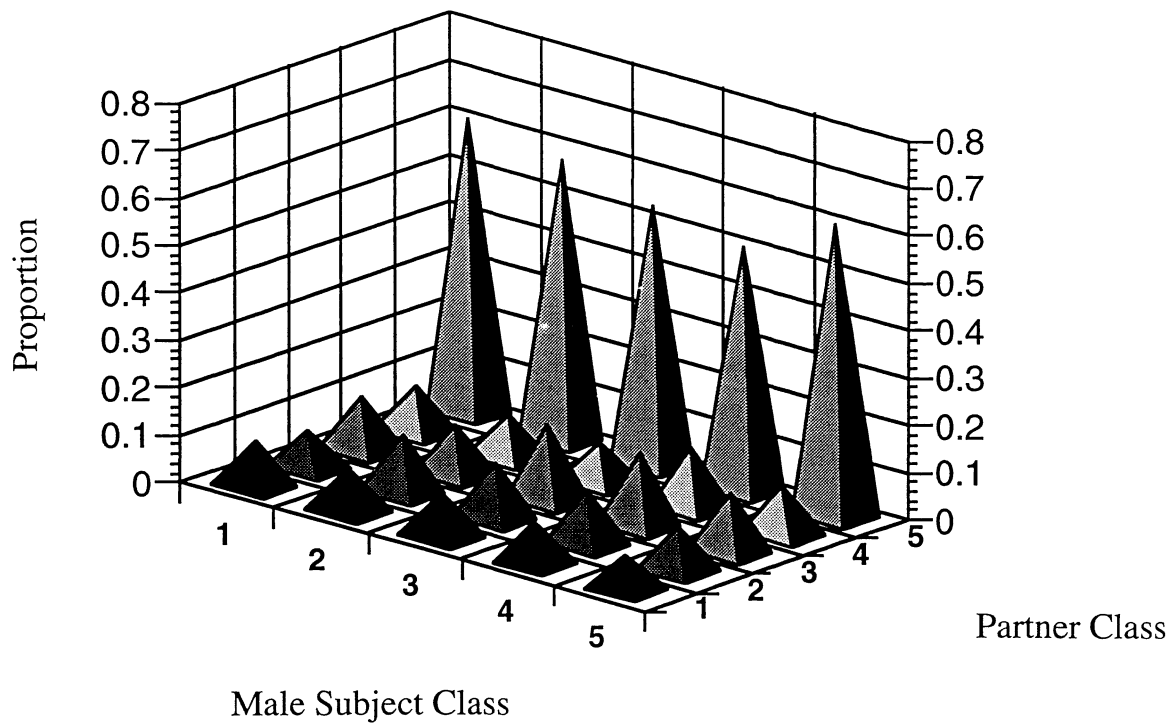


Figure 4 (a1)

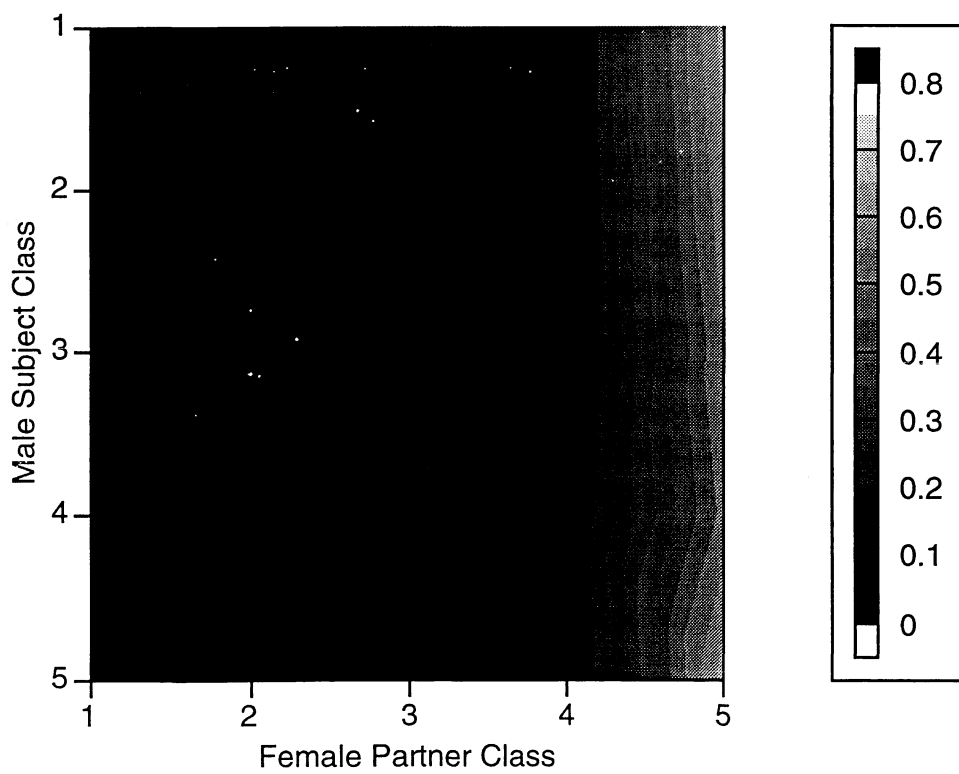


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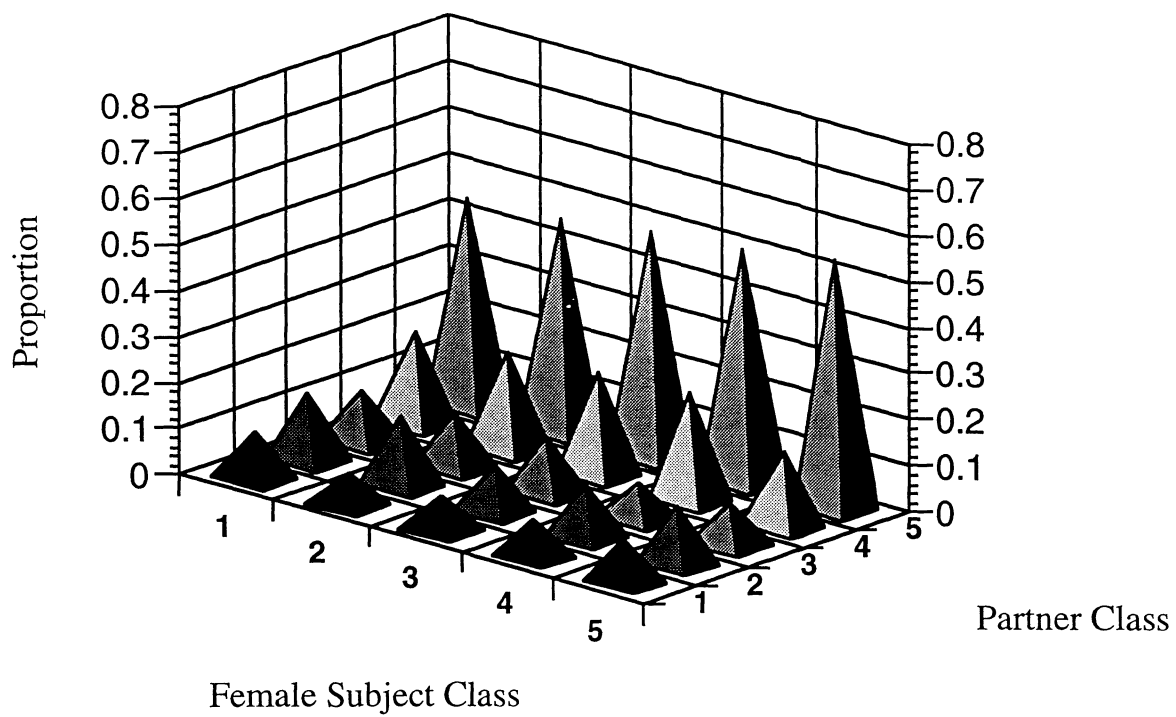


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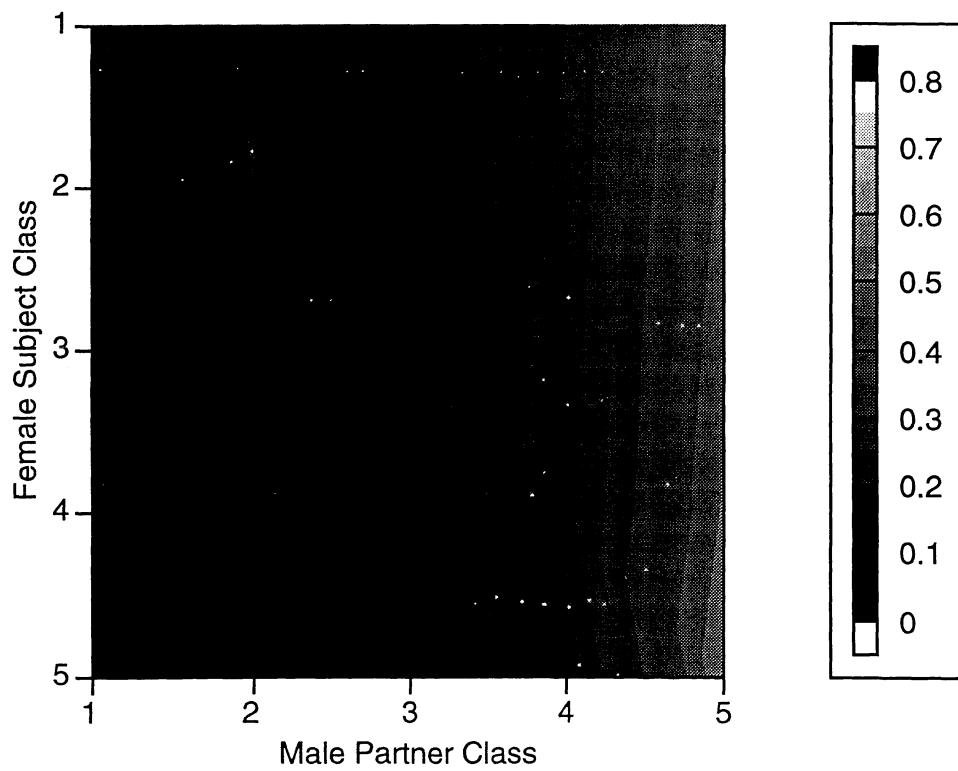


Figure 4 (b2)